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LETTER TO THE EDITOR

Replica symmetry breaking in weak connectivity systems

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Abstract. We propose a generalisation of Parisi's replica symmetry breaking scheme to spin glasses (or optimisation problems) described by a set of order parameters $Q^{(r)}_{\alpha_1 \ldots \alpha_r}$, or equivalently by a global order parameter $G\{\sigma_\alpha\}$, rather than by $Q^{(r)}_{\alpha_1\alpha_2}$ alone. We study the particular case of Derrida's p-spin model in the very dilute limit. The model is solved exactly in the large p limit and a freezing transition occurs. Using replicas and a single step of our replica symmetry breaking ansatz we recover the exact result. We discuss the implications for the global order parameter $G\{\sigma_\alpha\}$ when more than one step in the replica symmetry breaking process is taken.

1. Introduction

Consider dilute random bond systems with a bond probability distribution

$$\mathscr{P}(J_{ij}) = (1-\gamma)\delta(J_{ij}) + \gamma\rho(J_{ij}) \qquad (1)$$

where $(1-\gamma)$ is the fraction of absent bonds and for simplicity we keep to the spin glass with symmetrical ρ. Such spin systems are described in general by a set of order parameters $Q^{(r)}_{\alpha_1\ldots\alpha_r}$, one for each choice of r distinct replicas. These systems can be approached from two opposite limits.

(i) Strong connectivity $c = \gamma_z$ ($\gamma \sim 1$, z is the number of neighbours) and weak bonds with a characteristic bond strength $J \sim J_0/z^{1/2}$. As $z \to \infty$ (or $z = N$) one recovers the Sherrington and Kirkpatrick (1975, hereafter referred to as SK) limit. In that limit one only needs the order parameter $Q^{(r)}_{\alpha_1\alpha_2}$.

(ii) Weak connectivity $c = \alpha$ ($\gamma = \alpha/z$) and strong bonds J. As $z \to \infty$ the Viana and Bray (1985, hereafter referred to as VB) limit is recovered. In contrast, this limit still involves all $Q^{(r)}_{\alpha_1\ldots\alpha_r}$. If the bonds in turn become weak like $J \sim J_0(\alpha)^{1/2}$, $\alpha \to \infty$, the VB limit yields the same leading term as the SK limit.

Near T_c, in either of these limits one can describe the system with 'few' order parameters, viz $Q^{(2)}_{\alpha_1\alpha_2} \equiv \langle\sigma_{\alpha_1}\sigma_{\alpha_2}\rangle$ and $Q^{(4)}_{\alpha_1\ldots\alpha_4} \equiv \langle\sigma_{\alpha_1}\ldots\sigma_{\alpha_4}\rangle$ (or $Q^{(2)}$ alone after elimination of $Q^{(4)}$ via its equation of motion). Stability of the replica symmetric (RS) ansatz has been discussed in De Dominicis and Mottishaw (1986) with the condition

$$z < \tfrac{8}{3}(1 - 1/3\gamma) \qquad (2)$$

for RS stability. Note that the VB limit $\gamma \to 0$ is a worse starting point from the point of view of RS stability since there the right-hand side of (2) is always negative in that limit.

In optimisation problems or near $T = 0$ all $Q^{(r)}$ are equally important which has led Mézard and Parisi (1985) and Orland (1985) to introduce a global order parameter built with all the $Q^{(r)}$ of the problem. This approach has been extended to spin problems by De Dominicis and Mottishaw (1987a, b) by introducing the global order parameter $G\{\sigma_\alpha\}$

$$G\{\sigma_\alpha\} = z\sum_r b_r \sum_{(\alpha_1\ldots\alpha_r)} Q^{(r)}_{\alpha_1\ldots\alpha_r}\sigma_{\alpha_1}\ldots\sigma_{\alpha_r} \tag{3}$$

where, when $r \neq 0$,

$$zb_r = z\int_{-\infty}^{+\infty} \mathrm{d}u \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \ln\left|\int \mathrm{d}J\mathscr{P}(J)\exp(\beta Js)\right| \exp(us)(\tanh u)^r. \tag{4}$$

In the VB limit we have

$$zb_r = \alpha \int \mathrm{d}J\rho(J)(\tanh\beta J)^r \tag{5}$$

for r even (and $zb_r = 0$ for r odd), i.e. $b_{2r} > b_{2r+2}$, the 'masses' are ordered (which is not the case in general). In the RS case $G\{\sigma_\alpha\} = G(S \equiv \Sigma_\alpha\,\sigma_\alpha)$ satisfies the equation of motion

$$G(S) = z\int_{-\infty}^{+\infty} \mathrm{d}u\,\mathrm{d}v \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}x\,\mathrm{d}y}{(2\pi\mathrm{i})^2} \ln\left|\int \mathrm{d}J\mathscr{P}(J)\exp(\beta Jx)\right|$$
$$\times \exp(G(y))\exp(ux + vy)\exp[s\tanh^{-1}(\tanh u \tanh v)]. \tag{6}$$

In the VB limit (6) becomes, with $g(S) = \alpha + G(S)$ to keep previous notation,

$$g(S) = \alpha\exp(-\alpha)\int \mathrm{d}J\rho(J)\int_{-\infty}^{+\infty} \mathrm{d}v \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \exp[g(y) + vy]$$
$$\times \exp[s\tanh^{-1}(\tanh\beta J \tanh v)] \tag{7}$$

a simplified equation of motion independently derived by Mézard and Parisi (1987) and Kanter and Sompolinsky (1987). These authors also proposed a simple ansatz for (7) which was proved to be unstable by Mottishaw and De Dominicis (1987). A new ansatz containing (in some limit) a continuous component has been proposed by Wong *et al* (1988) for a related problem. However, it is not yet known whether this more elaborate ansatz is RS stable or not. In any case the question remains: how does one treat RS breaking when one has an infinite number of components $Q^{(r)}$ or a $g\{\sigma_\alpha\}$ function?

In this letter we consider first the simplest case where one has to break RS, namely spins interacting in p-plets ($p \to \infty$) as introduced by Derrida (1980, 1981)

$$H = -\sum_{(i_1\ldots i_p)} J_{i_1\ldots i_p}\sigma_{i_1}\ldots\sigma_{i_p}$$

with $J_{i_1\ldots i_p}$ a random coupling, taken here in the weak connectivity limit. In § 2 we briefly recall the results of Derrida in the strong connectivity ($\gamma = 1$) limit and the analysis of Gross and Mézard (1984) describing the spin-glass phase in terms of $Q^{(2)}_{\alpha_1\alpha_2}$. In § 3, we extend the results of Derrida for $\gamma \ll 1$ and give the exact result for the free energy. In § 4, we show how to describe the condensed phase in terms of the $Q^{(r)}_{\alpha_1\ldots\alpha_r}$. A one-step RS breaking scheme is proposed and shown to lead to the exact answer derived in § 3. In § 5 we describe RS breaking with two or more steps and its connection with the global order parameter $g\{\sigma\}$. In addition one explicitly obtains, when $\rho(J)$ is discrete, two distinct transtions (percolation and spin glass) at $T = 0$, a feature discussed most recently by Bray and Feng (1987).

2. Strong connectivity, weak bonds

This is Derrida's model with $\gamma = 1$, $J^2 = J_0^2 p!/N^{p-1}$.

Derrida has evaluated the probability to have M configurations with energies $E_1 \ldots E_M$, showing that, as $p \to \infty$, it factorises into

$$P(E_1, \ldots, E_M) = \prod_{a=1}^{M} P(E_a) \tag{8}$$

$$P(E) \sim \exp(-E^2/J_0^2 N). \tag{9}$$

This property enabled him to obtain the entropy and hence the free energy ($\beta \equiv 1/T$):

$$-\beta f = (\beta J_0)^2/4 + \ln 2 + \ln \cosh \beta h \tag{10}$$

for $T > T_c(h)$, where $T_c(h)$ is a de Almeida and Thouless (1978) line given by the condition of zero entropy:

$$s(T_c) = 0 = -(J_0^2/4T_c(h)) + \ln 2 + \ln \cosh(h/T_c(h)) - (h/T_c(h)) \tanh(h/T_c(h)). \tag{11}$$

For $T < T_c(h)$, in the frozen (spin-glass) phase, the free energy

$$f(T) = f(T_c(h)) \tag{12}$$

is frozen and the entropy remains null.

Approaching the problem in the standard replica fashion, Gross and Mézard (1984) were able to recover (10)-(12) and, at the same time, to give a description of the condensed phase in terms of the order parameter $Q^{(2)}_{\alpha_1\alpha_2}$ (and of a conjugate constraint variable $\lambda^{(2)}_{\alpha_1\alpha_2}$). The remarkable feature was that a one-step RS breaking, as in Parisi (1979), was enough to lead to the exact (10)-(12) result, with $Q^{(2)}_{\alpha_1\alpha_2}$taking two values (for $(\alpha_1\alpha_2)$ in the diagonal or off-diagonal blocks).

3. Weak connectivity, strong bonds: exact result

Here we now take $\gamma = \alpha p!/2N^{p-1}$ in the bond probability $\mathscr{P}(J_{i_1,\ldots,i_p})$.

As in § 2 we now evaluate $P(E_1 \ldots E_M)$ in the form

$$P(E_1 \ldots E_M) = \int \frac{d\hat{x}}{2\pi} R(\hat{x}) \prod_{a=1}^{M} Q(E_a, \hat{x}) \tag{13}$$

$$Q(E_a; \hat{x}) = \int \frac{d\hat{E}}{2\pi} \exp[i\hat{E}E_a - i\hat{x} \ln \cosh(iJ\hat{E})] \tag{14}$$

$$R(\hat{x}) = \int dx \exp\{2^N ix\hat{x} + \tfrac{1}{2}\alpha[\exp(2^N x) - 1]\} \tag{15}$$

for $N \gg p \gg 1$. From (13)-(15) one then derives in the ($p \to \infty$, $N \to \infty$) limit the free energy

$$-\beta f = \tfrac{1}{2}\alpha \ln \cosh \beta J_0 + \ln 2 + \ln \cosh \beta h \tag{16}$$

for $T > T_c(h; \alpha)$. Here we have used a discrete (symmetrical) distribution $\rho(J)$ localised at $\pm J_0$. In general, one has to replace the ln cosh of (16) (and (14)) by an average over $\rho(J)$, e.g. in (16)

$$\ln \cosh \beta J_0 \to \ln \int \rho(J) \exp(\beta J)\, dJ. \tag{17}$$

For $T < T_c(h; \alpha)$, i.e. below the de Almeida–Thouless line, the free energy remains frozen (and the entropy null).

Note that again if $J_0 \to J_0/\alpha^{1/2}$, $\alpha \to \infty$, one recovers Derrida's result (10).

In the zero-temperature limit (16) yields

$$s(T=0) = (\ln 2)(1 - \tfrac{1}{2}\alpha). \tag{18}$$

That is, for a discrete distribution, the entropy remains positive down to $T=0$ if

$$\alpha < \alpha_c = 2 \tag{19}$$

yielding a *spin-glass transition* for $\alpha_c = 2$ (a continuous distribution leads to a vanishing of the entropy at $T>0$). The *percolation transition* is easily established (e.g. via a geometrical argument) to occur at

$$\alpha_p = 2/p(p-1) \to 0 \tag{20}$$

in this model (cf the discussion in Bray and Feng (1987)).

4. Weak connectivity, strong bonds: replica approach

The free energy is easily written as

$$-\beta f n = -\sum_{r=2} \sum_{(\alpha_1 \ldots \alpha_r)} Q^{(r)}_{\alpha_1 \ldots \alpha_r} \mathrm{i}\lambda^{(r)}_{\alpha_1 \ldots \alpha_r} + \tfrac{1}{2}\alpha \sum_{r=2} b_r \sum_{(\alpha_1 \ldots \alpha_r)} (Q^{(r)}_{\alpha_1 \ldots \alpha_r})^p$$
$$+ \ln \tilde{Z} + \tfrac{1}{2}\alpha n \ln \cosh \beta J_0 + n \ln 2 \tag{21}$$

with b_r as in (5), and

$$\tilde{Z} = \operatorname*{Tr}_{\sigma} \exp\left(\sum_{r=2} \mathrm{i}\lambda^{(r)}_{\alpha_1 \ldots \alpha_r} \sigma_{\alpha_1} \ldots \sigma_{\alpha_r} \right) \tag{22}$$

where $\lambda^{(r)}$ is a constraint variable.

If one wants to introduce a one-step RS breaking (first step in the Parisi process), one writes $\alpha \equiv (K, \gamma)$ where K is the box number ($K = 1, 2, \ldots, n/m$) and γ the replica number in a box ($\gamma = 1, 2, \ldots, m$). The order parameter $Q^{(r)}_{\alpha_1 \ldots \alpha_r}$ is now characterised by the number of boxes with one spin (ν_1), the number of boxes with two spins (ν_2), etc, i.e. by a partition $\{\nu_l\}_r$ of r, such that

$$\sum_{l=1}^{r} \nu_l = r. \tag{23}$$

We write

$$Q^{(r)}_{\alpha_1 \ldots \alpha_r} \equiv Q^{(r)}_{\{\nu_l\}} \tag{24}$$

e.g. for $r=2$, one has $Q^{(2)}_{11} \equiv Q^{(2)}_{\nu_1=2}$ (the former off-diagonal component in the $Q_{\alpha\beta}$ matrix after one step of the Parisi process) and $Q^{(2)}_{2} \equiv Q^{(2)}_{\nu_2=1}$ (the former diagonal block component). Here and below we freely use two equivalent notations: the notation as in (24) and a more cumbersome, but more explicit, one where the partition is fully displayed as 11 or 2 for the two partitions of (2).

For $r=4$, one has $Q^{(4)}_{1111} \equiv Q^{(4)}_{\nu_1=4}$; $Q^{(4)}_{112} \equiv Q^{(4)}_{\nu_1=2,\nu_2=1}$; $Q^{(4)}_{13} \equiv Q^{(4)}_{\nu_1=1,\nu_3=1}$; $Q^{(4)}_{22} \equiv Q^{(4)}_{\nu_2=2}$; $Q^{(4)}_{4} \equiv Q^{(4)}_{\nu_4=1}$, corresponding to the five possible partitions of (4).

We need to define the spin functions

$$\Gamma^{(r)}_{\{\nu_i\}} \equiv \sum_{(K_1\gamma_1, K_2\gamma_2, \ldots, K_r\gamma_r)} \sigma_{K_1\gamma_1} \ldots \sigma_{K_r\gamma_r}\Bigg|_{\{\nu_i\}}$$

$$\equiv \Gamma^{(r)}_{\{\nu_i\}}(S_u) \tag{25}$$

where in (25) the summation is restricted to a given partition $\{\nu_i\}_r$ of the r spins into the n/m boxes K, i.e. satisfying (23). These turn out to be polynomials in the S_u $(u \leq r)$:

$$S_u = \sum_K (\sigma_K)^u \equiv \sum_K \left(\sum_\gamma \sigma_{K\gamma}\right)^u \tag{26}$$

e.g.

$$\Gamma^{(2)}_{\nu_1=2} = \sum_{(K_1 K_2)} \sigma_{K_1}\sigma_{K_2} = \tfrac{1}{2}[(S_1)^2 - S_2]. \tag{27}$$

The free energy then becomes

$$-\beta f n = -\sum_{r=2}\sum_{\{\nu\}} Q^{(r)}_{\{\nu\}} \mathrm{i}\lambda^{(r)}_{\{\nu\}}\Gamma^{(r)}_{\{\nu\}}(S_u = nm^{u-1}) + \tfrac{1}{2}\alpha \sum_{r=2} b_r \sum_{\{\nu\}} (Q^{(r)}_{\{\nu\}})^p \Gamma^{(r)}_{\{\nu\}}(S_u = nm^{u-1})$$

$$+\tfrac{1}{2}\alpha n \ln\cosh\beta J_0 + n\ln 2$$

$$+\ln\int\prod_u \mathrm{d}S_u \frac{\mathrm{d}\hat{S}_u}{2\pi}\exp\mathscr{L}(S_u; \mathrm{i}\hat{S}_u)$$

$$\times\left\{1 + \frac{n}{m}\left[\sum_{u=1}\mathrm{i}\hat{S}_{2u}(m)^{2u} + \ln 2\cosh\left(\beta hm + \sum_{u=1}\mathrm{i}\hat{S}_{2u-1}(m)^{2u-1}\right)\right]\right\} \tag{28}$$

$$\mathscr{L}(S_u; \mathrm{i}\hat{S}_u) = -\sum_u \mathrm{i}\hat{S}_u S_u + \sum_{r=2}\sum_{\{\nu\}} \mathrm{i}\lambda^{(r)}_{\{\nu\}}\Gamma^{(r)}_{\{\nu\}}(S_u). \tag{29}$$

From (28) and (29) via the use of the identity

$$\Gamma^{(r)}_{\{\nu_i\}}(S_{2u} = 0;\ S_{2u-1} = m^{2u-1}\,\mathrm{d}/\mathrm{d}X)\ln\cosh X$$

$$= \frac{(-1)^{r+1+\Sigma_{u=1}^{r/2}\nu_{2u}}}{r}\prod_{l=1}^{r}(C^l_m)^{\nu_l}[(\tanh X)^r - 1] \tag{30}$$

one derives the non-trivial stationarity condition

$$Q^{(r)}_{\{\nu\}} = (\tanh\beta hm)^{\Sigma_{u=1}^{r/2}\nu_{2u-1}}. \tag{31}$$

This recovers the Gross and Mézard result:

$$Q^{(2)}_{11} = \tanh^2\beta hm \tag{32}$$

$$Q^{(2)}_{2} = 1 \tag{33}$$

and extends it to all order parameters, e.g. $Q^{(4)}_{1111} = \tanh^4\beta hm$, $Q^{(4)}_{13} = Q^{(4)}_{112} = \tanh^2\beta hm$, $Q^{(4)}_{22} = Q^{(4)}_{4} = 1$, etc. Using (31) and the corresponding stationarity for $\lambda^{(r)}_{\{\nu\}}$ one obtains

$$-\beta f n = \tfrac{1}{2}\alpha\sum_{r=2} b_r \sum_{\{\nu\}}\Gamma^{(r)}_{\{\nu\}}(S_u = nm^{u-1})$$

$$+\tfrac{1}{2}\alpha n\ln\cosh\beta J_0 + n\ln 2 + (n/m)\ln 2\cosh\beta hm \tag{34}$$

i.e. after summation

$$-f=\frac{\alpha}{2(\beta m)}\ln\cosh(\beta mJ_0)+\frac{1}{\beta m}\ln 2\cosh\beta mh. \tag{35}$$

As in Gross and Mézard, the free energy is a function of the product βm and stationarity with respect to m is identical to zero entropy with $m=T/T_c(h;\alpha)$ in the spin-glass phase. At the boundary $m=1$ and (35) gives the free energy of the paramagnetic phase, thus recovering the results of § 3.

5. Two-step (or more) RS breaking

The parametrisation is easily inferred for two (or more) steps of RS breaking. The rule is that each new step introduces a new partitioning of the elements of the previous partition, e.g. if, at step one, we have (for $r=4$) the five partitions

(1111) (1; 3) (11; 2) (2 2) (4).

At step two they become

$$(1;3)\ \rightarrow \begin{pmatrix}1; & 3\\ & 111\end{pmatrix}\ \begin{pmatrix}1; & 3\\ & 11;2\end{pmatrix}\ \begin{pmatrix}1; & 3\\ & 3\end{pmatrix}$$

$$(11;2)\rightarrow \begin{pmatrix}11; & 2\\ & 11\end{pmatrix}\ \begin{pmatrix}11; & 2\\ & 2\end{pmatrix}$$

$$(2\ 2)\ \rightarrow \begin{pmatrix}2 & 2\\ 11 & 11\end{pmatrix}\ \begin{pmatrix}2 & 2\\ 2 & 11\end{pmatrix}\ \begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}$$

$$(4)\ \ \ \rightarrow \begin{pmatrix}4\\ 1111\end{pmatrix}\ \begin{pmatrix}4\\ 1;3\end{pmatrix}\ \begin{pmatrix}4\\ 11;2\end{pmatrix}\ \begin{pmatrix}4\\ 22\end{pmatrix}\ \begin{pmatrix}4\\ 4\end{pmatrix}.$$

Note that the number of independent r-order parameters (after step one) $N_\mu^{(1)}$ is given by the Euler generating function

$$\sum_r x^r N_r^{(1)}=\prod_{q=1}^{\infty}(1-x^q)^{-1}. \tag{36}$$

After step two

$$\sum_r y^r N_r^{(2)}=\prod_{q=1}^{\infty}(1-N_q^{(1)}x^q)^{-1} \tag{37}$$

with obvious generalisations.

The spin functions of § 4 are now (after step two) functions of $S_{u,v}$:

$$S_{u,v}=\sum_K\left[\sum_L\left(\sum_\gamma \sigma_{KL\gamma}\right)^v\right]^u \qquad \sigma_\alpha\equiv\sigma_{K,L,\gamma}. \tag{38}$$

The global order parameter $g(S)$ becomes $g\{S_u\}$ after step one and $g\{S_{u,v}\}$ after step two and the equations of motion replacing (6) or (7) are easily derived from the one for $G\{\sigma_\alpha\}$:

$$G\{\sigma_\alpha\}=z\operatorname*{Tr}_\tau\ln\left(\int dJ\mathscr{P}(J)\exp\left(\beta J\sum_\alpha\sigma_\alpha\tau_\alpha\right)\right)\exp(G\{\tau_\alpha\})/Z_G \tag{39}$$

$$Z_G=\operatorname*{Tr}_\tau\exp(G\{\tau\}) \tag{40}$$

or in the VB limit, with $g\{\sigma_\alpha\} = \alpha + G\{\sigma_\alpha\}$,

$$g\{\sigma_\alpha\} = \alpha \int \mathrm{d}J\rho(J) \operatorname*{Tr}_\tau \exp\left(\beta J \sum_\alpha \sigma_\alpha \tau_\alpha\right) \exp(g(\tau_\alpha))/Z_g. \qquad (41)$$

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